Effects of Atmosphere and Aircraft Motion on the Location and Intensity of a Sonic Boom

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In the present paper the problem of a shock propagating through a variable atmosphere is considered, and a rather complete treatment is presented. Techniques are given which permit the calculation of shock strength and location as a function of its initial configuration, the atmosphere through which it has propagated, and the distance it traveled. A specific application of this theory is made in considering the "sonic boom" caused by a supersonic aircraft. Problems such as complete acoustic refraction and/or accelerating aircraft, which give rise to shock configurations that are concave to the direction of propagation, are discussed. Attempts at solving these problems by acoustic techniques sometimes lead to physically unrealistic situations involving cusped shocks of high intensity. It is shown that when an approximation, better than the acoustic one, is used these difficulties are resolved.

Nomenclature

a = sound speed
$A(s)$ = ray-tube cross-section area
$B(s)$ = $(a_0 + w_0)^2 I(s) \{Ap_0/a_0\}^{1/2}$
c = Snell's constant
e = projected aircraft travel distance, in time Δt, along x axis
$E(s)$ = ray-tube energy-source term
$G(s)$ = gravity component along ray tube
h = altitude
$I(s)$ = $\exp\left\{\int_0^s \frac{w_{0s}\rho_0 - [(\gamma - 1)/2]w_0\rho_{0s}}{\rho_0(w_0 + a_0)} ds\right\}$
l = wave-front normal, x direction cosine
L = correction for wave-front position
m = wave-front normal, y direction cosine
M = Mach number
$M(s)$ = ray-tube mass-source term
n = wave-front normal, z direction cosine
$n_{i,j,k}$ = direction cosines of wave-front normal
ΔN = perpendicular distance between wave-front surfaces
Q = $(\phi_{x_i}{}^2)^{1/2}$
p = pressure
r = radial distance
R = body-shape factor
s = distance along ray
$S(\eta)$ = cross-section area of aircraft
t = time
t_0 = time for vertex ray to reach ground
U = wind speed in X direction
u = wind speed in x direction
u_0 = wind speed in x direction relative to speed at aircraft altitude
V = wind speed in Y direction
v = wind speed in y direction
v_0 = wind speed in y direction relative to speed at aircraft altitude
V_a = aircraft air speed
V_g = aircraft ground speed
V_s = shock speed
w = particle velocity along ray
x = ray coordinate
X = axis along direction of aircraft motion, moving with wind at aircraft altitude
X_f = axis along direction of aircraft motion, fixed
y = ray coordinate
Y = horizontal axis perpendicular to X, moving with wind at aircraft altitude
Y_f = horizontal axis perpendicular to X, fixed
z = vertical axis
Z = vertical axis, moving with wind at aircraft altitude
Z_f = vertical axis, fixed
γ = ratio of specific heats
λ = initial angle between wave front normal and x axis
μ = Mach angle
ν = angle between shock front and x axis
ϕ = position of wave front in space and time
Φ = angular measurement about aircraft axis
ρ = density
σ = position of shock front in space
θ = angle between x and X axes
ξ,η = distance along aircraft axis, measured from nose
η_0 = point on aircraft axis where last characteristic of expansion fan behind the shock leaves the body

Subscripts

f = fixed coordinate system
h = evaluated at initial, aircraft, altitude h
i,j,k = components in (i,j,k) direction
ph = physical variable, with dimensions
0 = atmospheric condition, lowest-order perturbation
1,2 = first-, second-order perturbation

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1. Introduction

A RATHER complete treatment of the sonic boom propagation problem will be presented in this paper. First, techniques will be given permitting shock-strength determination as a function of aircraft shape, altitude, Mach number, and atmospheric wind temperature and pressure variations. The shock-strength evaluation is based on a generalization of geometric acoustic ray-tube area concepts. Next, the acoustic ray-tracing equations are extended to describe shock propagation, and a method for determining the shock-ground intersection is given. Techniques developed are general enough that problems such as complete acoustic refraction and accelerating supersonic aircraft can be treated.

The first of these problems, complete acoustic refraction, occurs when a downward moving ray is refracted upward by atmospheric variations. At the point of horizontal slope, the wave front (normal to the ray) has a cusped shape, and two adjacent rays will cross. Attempts[1–3] at describing this situation by means completely dependent on acoustics can lead to some physically unrealistic results. Geometrical acoustic theory, which describes the wave amplitude (or shock strength) as being inversely proportional to the square root of the ray-tube cross-section area, will predict an infinite amplitude at points where rays cross, corresponding to zero-tube area. This is a physically unrealistic result; in fact, the use of acoustic theory which is predicated on small amplitude perturbations is highly questionable for this situation.

The second problem concerns the wave front caused by an accelerating supersonic aircraft. For this case acoustic theory shows the wave front to be concave to its direction of propagation and the rays, if extended far enough, to intersect. At the point of intersection, cusped shocks of infinite strength are predicted.[7, 8] The foregoing comments on the acoustic refraction problem, apply equally to this problem.

The key to handling both these problems is in treatment of the ray-tube area. Acoustic approaches always have the front propagating at local sound speed and the rays dependent on ambient atmospheric conditions. For the present approach the front propagates at shock speed, and a relation between the rays and the shock strength is obtained. By means of this relation it is shown that an increase in shock strength will cause the ray tube to diverge; this diverging, in turn, will inhibit further increases in strength until finally an equilibrium configuration is attained.

Whitham[4, 5] developed a theory, for describing real shock propagation, which includes first-order acoustic terms and second-order nonlinear terms. In Whitham's and other works[6–8] based on his, the assumption had been made that the disturbances are propagating through a uniform atmosphere. This restriction will be removed, and the solution to the varying atmosphere problem will be given in Sec. 2. Expressions will be derived there which show the dependence of shock strength on atmospheric conditions and distance traveled. These results then will be combined with Whitham's to relate the shock to aircraft speed and shape.

In Sec. 3 the acoustic ray-tracing equations, which are derived in the Appendix, and their relation to the aircraft coordinate system is discussed. These equations are then altered to include true shock-propagation speeds instead of acoustic speeds. In addition, a technique is presented for determining the locus of the shock-ground intersection.

The ray-tube area is discussed in Sec. 4. An expression is developed which reduces to Whitham's[5] for steady flight in a uniform atmosphere, and which reduces to Rao's[7] for accelerating flight in a uniform atmosphere. This expression is more general than that used by either of these two authors in that it includes a term that gives the effect of shock-propagation speed on ray-tube area.

In Refs. 10 and 11, Whitham uses an expression relating shock strength (∼propagation speed) and ray-tube area. He shows that converging rays, such as are associated with a concave propagating shock, will cause the ray-tube area to decrease. This decrease will in turn induce a stronger shock that propagates faster. The faster propagation of the concave part of the shock will tend to flatten the shock shape until a stable configuration is attained. Whitham's theory involves disturbances propagating along the shock front; this, however, cannot be included in the present ray-tube analysis since the basic assumption here is that the shock propagates down each tube independent of the adjoining tubes. The result of the present theory is, however, similar to Whitham's in that as the tube area decreases the shock strength will increase, which will then cause the rays to diverge. This divergence induces an increase in tube area until finally an equilibrium between the tube area and shock strength is attained.

One of the main difficulties in presenting this theory is the interdependence between the three parts of the problems; shock strength, shock location, and ray-tube area. It is hoped that the development is reasonably logical and that the cross referencing within the paper does not prove too distracting. A brief summarizing outline of the results will be given in Sec. 5.2.

2. Shock Strength

2.1 Shock Strength and Atmospheric Variations

Assume the shock is propagating through an atmosphere in which there may be pressure, density, sound speed, and wind variations with altitude but not with time. Two frames of reference will be used in this problem. The first is a moving reference frame in which the coordinate system travels with the wind at aircraft altitude. As far as the shock is concerned, its strength will be affected by the gradients of wind, temperature, and density relative to where the shock starts. Consider an aircraft moving at a given Mach number and altitude in a uniform still atmosphere and again in a uniform moving atmosphere; the strength of the shock as it reaches the ground will be the same for both cases. However, to an observer on the ground (in a fixed reference frame) the total shock distance traveled will be different. This difference is due, in the one case, to convection of the shock by the uniform wind. In computing the shock strength as a function of distance traveled one would expect that the further the shock propagates from its source, the greater will be its attenuation. However, in the problem just posed the two shocks travel different distances but still must have the same strength upon arrival at the ground. This apparent paradox is resolved by measuring shock travel distance relative to a coordinate system moving with the wind at aircraft altitude.

The second frame of reference is fixed with respect to the ground and is used only when the shock-ground intersection is computed, in Sec. 3.3. The authors therefore will assume, unless otherwise indicated, that the coordinate system is moving with the wind, and, hence, all velocities are relative to the wind velocity at aircraft altitude.

The equations for conservation of mass, momentum, and energy along a ray tube are

$$\begin{aligned} \rho_t + w\rho_s + \rho w_s + (\rho w A_s/A) &= M(s) \\ w_t + ww_s + (1/\rho)p_s &= G(s) \\ (p_t + wp_s) - (\gamma p/\rho)(\rho_t + w\rho_s) &= E(s) \end{aligned} \tag{2.1}$$

Here $\rho, w, p, A = A(s), s$ are density, particle velocity along the ray, pressure, ray-tube cross-section area, and distance along the ray. $M(s)$ and $E(s)$ are mass and energy source terms, and $G(s)$ is the component of gravity along the ray. The variables are assumed to be dimensionless; their relation to physical variables is as follows:

$$s_{\text{ph}} = sh \qquad t_{\text{ph}} = ht/a_h$$

$$\rho_{\text{ph}} = \rho\rho_h \qquad w_{\text{ph}} = a_h w \qquad p_{\text{ph}} = \rho_h a_h^2 p$$

Constants ρ_h and a_h are density and sound speed evaluated at altitude h.

Within the present theory the shock propagates down a ray tube perpendicular to the sides of the tube. Hence the flow within the tube, induced by shock motion, will remain inside the tube provided there is no gradient in the cross wind; for this case mass and energy flux through a ray tube, $M(s)$ and $E(s)$ are both zero. When there is a cross wind, $M(s)$ and $E(s)$ are not zero, their form being quite complicated since they involve derivatives of the flow variables in the direction normal to the ray.

The ray-tube area term, A_s, was shown by Whitham[5] (see also Sec. 4 1) to be proportional to distance, s, along the ray for uniform flight in a uniform atmosphere. He then generalized this definition[5] to account for an accelerating aircraft. This will be generalized still further in Sec. 4, to account for a varying atmosphere as well as aircraft acceleration.

The quantities $E(s)$ and $M(s)$ can be simplified by noting that they represent mass and energy source terms, i.e., mass and energy being convected into a ray tube by the wind. This mass and energy will consist of atmospheric plus perturbation terms. For the present theory, only the convected atmospheric terms will be included. Neglecting the convected perturbation terms is in keeping with the assumption that the propagation of the disturbance down each ray tube can be treated separately. Since there is no mass or energy created, the quantities $E(s)$ and $M(s)$ are equated to the zeroth-order [see Eqs. (2.2)] atmospheric terms on the left-hand side of Eqs. (2.1), insuring that atmospheric mass and energy are conserved.

The solution to Eqs. (2.1) will be assumed to take the form of a perturbation on atmospheric conditions:

$$\begin{aligned} p &= p_0 + p_1(t-\sigma) + p_2(t-\sigma)^2 + \ldots \\ \rho &= \rho_0 + \rho_1(t-\sigma) + \rho_2(t-\sigma)^2 + \ldots \\ w &= w_0 + w_1(t-\sigma) + w_2(t-\sigma)^2 + \ldots \end{aligned} \quad (2.2)$$

In Eqs. (2.2) atmospheric terms are zeroth order and are assumed to depend only on distance s. The amplitudes p_1, p_2, ρ_1, etc. are to be determined; they also depend only on s. Time dependence is introduced through the function $(t - \sigma)$. The quantity σ, a function of s, is equal to s/a_0 in a uniform atmosphere; however, it is unknown in a nonuniform atmosphere. Curves $(t - \sigma) =$ const give the positions of the wave front in space. The form given in Eqs. (2.2) is valid for small values of $(t - \sigma)$, i.e., for points near the wave front (the scaling is assumed to be such that $t = 0$ corresponds to s or $\sigma = 0$).

Derivatives of the functions in Eqs. (2.2) take the following form

$$p_t = p_1 + 2p_2(t-\sigma) + \ldots$$

$$p_s = p_{0s} + p_{1s}(t-\sigma) - \sigma_s[p_1 + 2p_2(t-\sigma)] + \ldots \text{etc.}$$

Substituting these into the energy equation one obtains for lowest-order terms

$$w_0p_{0s} + p_1(1 - w_0\sigma_s) - (\gamma p_0/\rho_0)[w_0\rho_{0s} + \rho_1(1 - w_0\sigma_s)] = E(s)$$

or

$$\begin{aligned} w_0[p_{0s} - (\gamma p_0/\rho_0)\rho_{0s}] &= E(s) \\ p_1 &= (\gamma p_0/\rho_0)\rho_1 \end{aligned} \quad (2.3)$$

The quantity $E(s)$ vanishes either if $w_0 = 0$ or if the atmosphere is isentropic, $p_0 \sim \rho_0^{\gamma}$.

Using Eqs. (2.2) and the results of Eqs. (2.3), one obtains for the mass and momentum equations, respectively, zeroth-, first-, and second-order equations:

$$\begin{aligned} & w_0p_{0s} + \gamma p_0\left(w_{0s} + w_0\frac{A_s}{A}\right) = \frac{\gamma p_0}{\rho_0}M(s) + E(s) \\ & p_1(1 - w_0\sigma_s) - \gamma p_0 w_1\sigma_s = 0 \\ & 2p_2(1 - w_0\sigma_s) - 2\gamma p_0 w_2\sigma_s + w_0p_{1s} + \gamma p_0 w_{1s} + w_1\left[p_{0s} - p_1\sigma_s(\gamma+1) + \left(\frac{\gamma p_0 A_s}{A}\right)\right] = \frac{\gamma p_0}{\rho_0}\left(\frac{p_1}{p_0} - \frac{\rho_1}{\rho_0}\right)M(s) \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} & w_0w_{0s} + (1/\rho_0)p_{0s} = G(s) \\ & w_1(1 - w_0\sigma_s) - (p_1/\rho_0)\sigma_s = 0 \\ & 2w_2(1 - w_0\sigma_s) - (2/\rho_0)p_2\sigma_s + w_1(w_{0s} - w_1\sigma_s) + w_0w_{1s} + (1/\rho_0)\,[p_{1s} - (p_1/\gamma p_0)(p_{0s} - p_1\sigma_s)] = 0 \end{aligned} \quad (2.5)$$

The zeroth-order equation in (2.4) defines $M(s)$

$$M(s) = w_0\rho_{0s} + \rho_0 w_{0s} + w_0\rho_0 A_s/A$$

The first-order equations in (2.4) and (2.5) are homogeneous simultaneous equations for w_1 and p_1. In order to have a nonzero solution the determinant of coefficients must be zero, i.e.,

$$(\gamma p_0/\rho_0)\sigma_s^2 - (1 - w_0\sigma_s)^2 = 0 \quad (2.6)$$

or

$$\sigma_s = [\pm 1/(a_0 \pm w_0)]$$

[The first equation in (2.6) corresponds to the eikonal equation of optics.] Take the plus sign in Eqs. (2.6) since this represents outgoing waves, that is, waves propagating in the $+s$ direction. Using Eqs. (2.6) one has

$$p_1 = (\gamma p_0/a_0)w_1 \quad (2.7)$$

and, differentiating, this gives

$$p_{1s} = \frac{\gamma p_0}{a_0}w_{1s} + \frac{\gamma w_1}{a_0}\left(p_{0s} - \frac{p_0 a_{0s}}{a_0}\right) \quad (2.8)$$

The quantities p_2 and w_2 can be eliminated by multiplying the third equation in (2.5) by $\gamma p_0/a_0$, adding it to the third equation in (2.4), and using (2.6). After substitution of p_1 in terms of w_1 by using Eqs. (2.7) and (2.8), the resulting differential equation is

$$2w_{1s} + w_1\left[\frac{A_s}{A} + \frac{p_{0s}}{p_0} - \frac{a_{0s}}{a_0} + \frac{2w_{0s}\rho_0 - (\gamma-1)w_0\rho_{0s}}{\rho_0(w_0 + a_0)}\right] - \frac{(\gamma+1)w_1^2}{(w_0+a_0)^2} = 0 \quad (2.9)$$

Before integrating Eq. (2.9) some comments on the theory and results can be made. The lowest-order perturbation relation is that given in Eq. (2.7); this corresponds to acoustic or weak shock theory. The next result is obtained by omitting the nonlinear w_1^2 term in Eq. (2.9); this corresponds to the theory of geometrical acoustics which is the next order (but not nonlinear) improvement. It can be shown that this equation agrees with Eq. (56) of Ref. 9 to terms $0(w_0^2/a_0^2)$. For an isothermal atmosphere with no winds, $a_{0s} = w_0 = 0$, and Eq. (2.9) (neglecting w_1^2) integrates to $w_1(Ap_0)^{1/2} =$ const. Using Eq. (2.7), this can be written in the form now commonly used to give a correction for varying atmospheric pressure to sonic boom strength estimates

$$p_1 = p_{1h}(p_0/p_{0h})^{1/2}(A_h/A)^{1/2}$$

By retaining the w_1^2 term in Eq. (2.9) an improvement to geometrical acoustics is obtained. This is the best that can be done without involving entropy losses, which are third-order perturbation effects. Equation (2.9) can be integrated after introducing the function $I(s)$, where

$$I(s) = \exp\left\{\int_0^s \frac{w_{0s}\rho_0 - [(\gamma-1)/2]w_0\rho_{0s}}{\rho_0(w_0+a_0)}\,ds\right\}$$

the resulting integral is

$$w_1 = \frac{-2\left\{\dfrac{a_0(s)}{A(s)p_0(s)}\right\}^{1/2}}{(\gamma+1)I(s)\displaystyle\int_0^s \left\{\frac{a_0(s')}{A(s')p_0(s')}\right\}^{1/2} \frac{ds'}{I(s')[w_0(s')+a_0(s')]^2}} \quad (2.10)$$

Equation (2.10) relates the perturbation strength to distance along the ray. As the shock moves from the aircraft the cumulative effect of the expansion wave behind the shock wave is felt. It is seen[4, 5] that this is what causes the attenuation represented by the integral in Eq. (2.10). The expansion wave, in turn, is dependent on body shape. In the next section the relation between shock strength and aircraft body shape will be determined.

2.2 Shock Strength and Aircraft Shape

For acoustical theory the wave-front position can be given by $t - \sigma(s) = \text{const}$; however, it is possible, within the present improved theory to obtain a better prediction of shock position. Assume the shock location along a ray can be given by $t - \sigma(s) = -L(s)$. The quantity $L(s)$ is the correction of the present theory over acoustic theory. If one lets V_s denote shock velocity

$$1/V_s = (dt/ds) = (d\sigma/ds) - (dL/ds) = [1/(a_0 + w_0)] - (dL/ds) \quad (2.11)$$

using (2.6). Another expression for V_s is obtained by using the fact that, to the present order of approximation, the shock speed is the average of the propagation speeds in front of and behind the shock:[12]

$$\begin{aligned} V_s &= \tfrac{1}{2}(w_0 + a_0 + w + a) = w_0 + a_0 + \tfrac{1}{2}(w_1 + a_1)(t - \sigma) \\ &= w_0 + a_0 - \tfrac{1}{2}(w_1 + a_1)L(s) \\ &= w_0 + a_0 - [(\gamma+1)/4]w_1L(s) \end{aligned} \quad (2.12)$$

$$\frac{p - p_0}{p_0} = \frac{\dfrac{2^{3/4}\gamma}{(\gamma+1)^{1/2}}\dfrac{M^{3/4}}{(M^2-1)^{1/4}}\left\{\displaystyle\int_0^{\eta_0}\frac{1}{2\pi}\int_0^{\eta}\frac{S''(\xi)d\xi}{(\eta-\xi)^{1/2}}d\eta\right\}^{1/2}}{a_0I(s)\left[\dfrac{Ap_0}{a_0}\right]^{1/2}\left[\displaystyle\int_0^s\frac{ds}{B(s)}\right]^{1/2}h^{3/4}} \quad (2.16)$$

or

$$1/V_s = [1/(w_0 + a_0)] + [(\gamma+1)/4][w_1L/(w_0 + a_0)^2]$$

In the foregoing equation the quantity a_1 was eliminated by using the weak shock identity [corresponding to Eq. (2.7)]:

$$a_1 = [(\gamma - 1)/2]w_1$$

Equating Eqs. (2.11) and (2.12), and using Eq. (2.10)

$$2\frac{dL}{ds} = \frac{L}{B(s)\displaystyle\int_0^s \frac{ds}{B(s)}}$$

with

$$B(s) = (a_0 + w_0)^2 I(s)\left[\frac{A(s)p_0(s)}{a_0(s)}\right]^{1/2}$$

This is integrated to yield

$$L(s) = R\left[\int_0^s \frac{ds}{B(s)}\right]^{1/2} \qquad R = \text{const} \quad (2.13)$$

Using the first-order relations in Eqs. (2.2) and (2.7), and then substituting Eqs. (2.10) and (2.13) one obtains an expression giving the pressure jump across the shock, relative to local pressure,

$$\begin{aligned}\frac{p - p_0}{p_0} &= \frac{\gamma w_1}{a_0}(t - \sigma) = \frac{\gamma w_1}{a_0}[-L(s)] \\ &= \frac{2\gamma R/(\gamma+1)}{a_0 I(s)\left[\dfrac{Ap_0}{a_0}\right]^{1/2}\left[\displaystyle\int_0^s\frac{ds}{B(s)}\right]^{1/2}}\end{aligned} \quad (2.14)$$

The constant R, which contains the aircraft body-shape factor, is determined by comparing Eq. (2.14) to its counterpart, Eq. (13) of Ref. 5. This equation is

$$\frac{p_1 - p_0}{p_0} = \frac{K}{A^{1/2}\left(\displaystyle\int_0^s \frac{ds}{A^{1/2}}\right)^{1/2}} \quad (2.15)$$

Using Eqs. (54) and (13) of Ref. 5 (the γ in the former equation should be in the numerator),

$$\begin{aligned} K &= \left[\frac{4\gamma a_0}{\gamma+1}\int_0^{T_0} F(T')dT'\right]^{1/2} \\ &= \left\{\frac{4\gamma a_0}{\gamma+1}\cdot\frac{\gamma M^{2.5}}{2^{1/2}(M^2-1)^{1/2}} \times \int_0^{T_0}\frac{1}{2\pi}\int_0^{UT'}\frac{S''(\xi)d\xi}{(UT'-\xi)^{1/2}}dT'\right\}^{1/2} \\ &= \frac{2^{3/4}\gamma}{(\gamma+1)^{1/2}}\frac{M^{3/4}}{(M^2-1)^{1/4}}\left[\int_0^{\eta_0}\frac{1}{2\pi}\int_0^{\eta}\frac{S''(\xi)d\xi}{(\eta-\xi)^{1/2}}d\eta\right]^{1/2}\end{aligned}$$

Equation (2.15) was derived assuming a uniform atmosphere. To reduce the results to this case, set $p_0 = \rho_0 = a_0 = 1$ and $w_0 = 0$ on the right-hand side of Eq. (2.14), obtaining

$$[2\gamma/(\gamma+1)]R = Kh^{-3/4}$$

The factor $h^{-3/4}$ is to make the double integral dimensionless. Combining the forementioned results, one has

Equation (2.16) gives the pressure jump across the shock, in a nonuniform atmosphere, as a function of distance traveled, aircraft shape, and atmospheric variations.

In order to determine the wind component w_0 along the ray, the position as well as the slope of the ray corresponding to distance s must be known. This will involve a simultaneous solving of the ray equations (given in the next section) and the shock-strength equation, (2.16). The exact expression for w_0 is given in Eqs. (3.2) or (3.11). However, since wind variation is small in comparison to sound-speed magnitude, a simple approximation for w_0 can be made. One could, for example, assume the ray to be a straight line from its source to its destination and then let w_0 be the wind component along this line. The ray-tube area term A is yet to be defined. This term will be discussed in Sec. 4.1.

3. Ray Tracing and Shock Location

3.1 Acoustic Equations

It will be assumed that atmospheric sound speed a_0 and the horizontal winds (u_0,v_0) in the (x,y) directions are functions of height z alone; the vertical wind component will be neglected. If the coordinate system moves with the wind at aircraft altitude and is so aligned that the normal to the wave front is parallel to the (x,z) plane, the acoustic ray equations (derived in Appendixes A.1 and A.3) are

$$\frac{dx}{dz} = \frac{la_0 + u_0}{na_0} \qquad \frac{dy}{dz} = \frac{v_c}{na_0} \qquad \frac{dt}{dz} = \frac{1}{na_0}$$

$$\frac{ds}{dz} = -\left[\left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2 + 1\right]^{1/2} \qquad l = \frac{a_0}{c - u_0}$$

$$n = -(1 - l^2)^{1/2} \qquad c = \frac{a_h}{l_h} \tag{3.1}$$

The wind component along the ray now can be determined,

$$w_0 = u_0\frac{dx}{ds} + v_0\frac{dy}{ds} = \left(u_0\frac{dx}{dz} + v_0\frac{dy}{dz}\right)\frac{dz}{ds}$$
$$= \frac{u_0(dx/dz) + v_0(dy/dz)}{[(dx/dz)^2 + (dy/dz)^2 + 1]^{1/2}} \tag{3.2}$$
$$= \frac{u_0la_0 + u_0^2 + v_0^2}{[a_0^2 + u_0^2 + v_0^2 + 2u_0la_0]^{1/2}}$$

Geometric identities will be developed now by means of which the shock and ray cone, occurring for any physical problem, will be related to the ray-tracing equations (3.1). What is required, for any selected ray, is the position of the (x,y,z) coordinates of Eqs. (3.1) relative to the (X,Y,Z) aircraft-coordinate system. The (x,y,z) shock-coordinate system has wind velocities (u_0, v_0) in the (x,y) directions; since this coordinate system moves with the wind at the aircraft altitude, the velocities (u_0,v_0) relative to the system vanish at this altitude. The air-speed vector is along the negative X axis.

It is assumed that data are desired for rays spaced at angles Φ (see Fig. 1) about the axis of rotation (X axis). The ray-cone angle ($=90° - \mu$ where μ = Mach angle) is set by the flight condition. All data are determined in terms of μ and Φ. θ is the angle the wind components must be rotated (about the Z axis) in order to be lined up with the (x,y) coordinates, to which Eqs. (3.1) refer. Also, since rotation is about the Z axis, $z = Z$.

From the definition of μ, $\tan(90° - \mu) = (M^2 - 1)^{1/2}$, hence one has, from Fig. 1,

$$S = R\sin\Phi \qquad N = R/\tan(90° - \mu) \qquad \tan\theta = S/N$$

$$\tan\theta = \tan(90° - \mu)\sin\Phi = (M^2 - 1)^{1/2}\sin\Phi$$

therefore,

$$\cos\theta = \frac{1}{[1 + (M^2 - 1)\sin^2\Phi]^{1/2}}$$
$$\sin\theta = \frac{(M^2 - 1)^{1/2}\sin\Phi}{[1 + (M^2 - 1)\sin^2\Phi]^{1/2}} \tag{3.3}$$

Let λ be the initial angle between the ray and the positive x axis; then $l_h = \cos\lambda$. The Mach number and ray angle are related to l_h as follows:

$$T = R\cos\Phi \qquad P = S/\sin\theta = R\sin\Phi/\sin\theta$$

$$\tan\lambda = T/P = \cot\Phi\sin\theta$$
$$= \cos\Phi(M^2 - 1)^{1/2}/[1 + (M^2 - 1)\sin^2\Phi]^{1/2}$$

$$l_h = \cos\lambda = (-1/M)[1 + (M^2 - 1)\sin^2\Phi]^{1/2} \tag{3.4}$$

Letting u_0 and v_0 be the relative wind components along the x and y axes, one has

$$u_0 = (U - U_h)\cos\theta + (V - V_h)\sin\theta$$
$$v_0 = -(U - U_h)\sin\theta + (V - V_h)\cos\theta \tag{3.5}$$

where $\sin\theta$ and $\cos\theta$ are given in Eqs. (3.3).

Snell's constant is

$$c = a_0/l \qquad \text{evaluated at the initial point}$$
$$= a_h/\cos\lambda = -a_hM\cos\theta \tag{3.6}$$

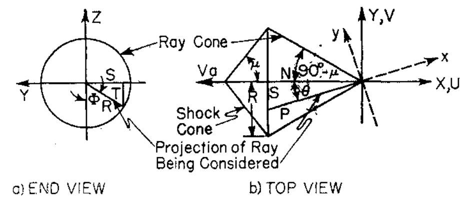

Fig. 1 Initial ray-cone coordinates

3.2 Improving the Acoustic Equations

In keeping with the theory as a whole, several parameters in Eqs. (3.1) will be improved; however, the form of the equation will be retained. First the local sound-speed term a_0 will be replaced by the shock-propagation speed V_s. This substitution can be used in both Eqs. (3.1) and (3.2).

The use of shock-propagation speed instead of sound speed leaves the derivation of Snell's law, as given in the Appendix, open to question. It will be shown now that a plane-propagating shock satisfies the same refractive law. Consider a shock propagating through a region in which atmospheric properties (Fig. 2) on either side of some horizontal line are uniform but different. The component along the x axis (Fig. 2) of the incident shock velocity relative to the wind must equal that of the refracted shock. That is both sections of the shock travel along the x axis at the same speed. Therefore,

$$(V_{1s}/\sin\nu_1) - u_1 = (V_{2s}/\sin\nu_2) - u_2 \tag{3.7}$$

This relation, which corresponds to Snell's law, will be taken to hold all along the path of shock propagation.

If one assumes now that the initial shock angle is prescribed by the Mach angle, one has, after making the identification $\sin\nu = -l$,

$$\frac{V_s}{l} + u_0 = \left(\frac{V_s}{l} + u_0\right)_{\text{initial}} = -a_hM\cos\theta$$
$$= -V_a\cos\theta \tag{3.8}$$

To obtain this relation Eqs. (3.3), (3.4) have been used, and, at the initial Mach cone, $u_h = 0$, $V_{sh} = a_h$. One sees then that, for any direction, the shock propagates at the same speed as the component of aircraft velocity in that direction. Eq. (3.8) can be derived directly from Eq. (3.6) by simply replacing sound speed by shock speed. In addition, the accuracy of Eq. (3.8) can be improved by using some of Whitham's[4] results relating the initial shock properties to the body slope at the nose.

In any case, Eq. (3.8) is of considerable importance in that it gives a relation between the ray angle and the shock velocity (or strength) V_s, as well as the ambient atmospheric and initial conditions,

$$l = \frac{-V_s}{u_0 + V_a\cos\theta} = -\sin\nu \tag{3.9}$$

This relation will be used in Sec. 4.1 for determining an expression for ray-tube area.

Equations (3.1) and (3.2) can be rewritten with shock propagation speed instead of sound speed, and with direction cosines just defined:

$$\frac{dx}{dz} = \frac{\rho V_s + u_0}{nV_s} \qquad \frac{dy}{dz} = \frac{v_0}{nV_s} \qquad \frac{dt}{dz} = \frac{1}{nV_s}$$
$$\frac{ds}{dz} = -\left[\left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2 + 1\right]^{1/2} \tag{3.10}$$

$$l = \frac{-V_s}{u_0 + V_a\cos\theta} = -\sin\nu \qquad n = -(1 - l^2)^{1/2} = -\cos\nu$$

$$w_0 = \frac{u_0lV_s + u_0^2 + v_0^2}{\{V_s^2 + u_0^2 + v_0^2 + 2u_0lV_s\}^{1/2}} \tag{3.11}$$

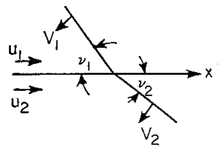

Fig. 2 Shock refraction through a nonuniform region

These equations must be integrated simultaneously with the shock strength Eq. (2.16). Equations (3.10) furnish the shock location and propagation distance s, while Eq. (2.16) furnishes the shock strength for determining V_s. Shock propagation speed is related to pressure jump as follows:

$$V_s = a_0 \left[1 + \frac{\gamma + 1}{4\gamma} \left(\frac{p - p_0}{p_0} \right) \right] \tag{3.12}$$

It should be noted that the V_s used here, and in the remainder of this paper, is different from that used in Eqs. (2.11) and (2.12). In Sec. 2, V_s represented shock velocity relative to a fixed point, $s = 0$. The V_s used here is shock-propagation speed relative to local wind. The present V_s equals the one of Sec. 2 minus w_0.

3.3 Shock-Ground Intersection

If Eqs. (3.10) are integrated from $z = 0$ to $z = -h$ (i.e., for an aircraft at altitude h) for a given angle Φ, a point on the ray-ground intersection is obtained. This must be related to a fixed coordinate system before the shock-ground locus can be constructed. Two simple transformations are required for this; the ray coordinates [Eqs. (3.10)] are related to the (X,Y,Z) coordinates by a rotation, and these coordinates are related to the fixed coordinates by a translation.

The (X,Y,Z) system initially coincides with the fixed system (X_f,Y_f,Z_f), its negative X axis aligned with the airspeed vector. For later times this system moves away from the fixed system at aircraft altitude wind speed (Fig. 3a).

These two systems are related as follows:

$$\begin{aligned} X_f &= X + U_h t \\ Y_f &= Y + V_h t \end{aligned} \tag{3.13}$$

Where t is the time taken by the selected ray to reach the ground and (U_h, V_h) are wind components along (X_f, Y_f), at the airplane altitude.

The relation between the ray system and the (X,Y,Z) system as indicated in Fig. 1b is

$$\begin{aligned} X &= x \cos\theta - y \sin\theta \\ Y &= x \sin\theta + y \cos\theta \end{aligned} \tag{3.14}$$

Equations (3.13) and (3.14), when combined, will give the locus of the ray-ground intersection. This is the locus of disturbances that left the aircraft at the same instant. What is desired, however, is the shock locus, i.e., disturbances that arrive at the ground at the same instant. This is determined easily for an aircraft flying at constant velocity, for in this case the shock locus is invariant with time. Consider first the shock and ray intersections that touch at a common vertex (Fig. 4). Points on the shock to either side of the vertex corresponds to rays that took a longer time to reach the ground. The shock moves along the ground at aircraft ground speed; a sequence of shock positions is shown in Fig. 4.

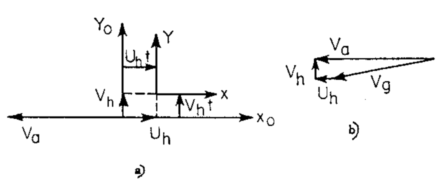

Fig. 3 a) The moving and fixed coordinate systems; b) air-speed and ground-speed vector relation

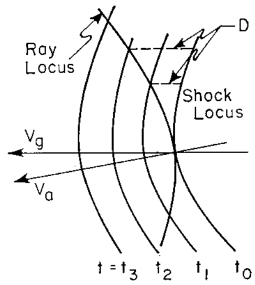

Fig. 4 Ray-shock-ground intersection. The shock is symmetric about the V_a axis but is displaced in the V_g direction

If for each point of the ray-ground intersection the distance D [Eq. (3.15) below] is projected back parallel to the ground speed direction, the corresponding point on the original $t = t_0$ shock is obtained:

$$\begin{aligned} D &= V_g(t - t_0) \\ V_g &= \text{aircraft ground speed} = [(V_a - U_h)^2 + V_h^2]^{1/2} \end{aligned} \tag{3.15}$$

t_0 = time for vertex ray to reach ground
t = time for selected ray to reach ground

The relation between ground speed and air speed, head wind and side wind is as indicated in Fig. 3b. Ray travel times are obtained from integration of Eqs. (3.10).

4. Ray Tube Area

4.1 Ray Distance, Atmospheric Variations, and Aircraft Acceleration

In a uniform atmosphere the shock is cone shaped, and the ray-tube cross-section area is $A = 2\pi rd$ (see Fig. 5). For the ray directly below the aircraft, $\theta = 0$ and $rd = re \cos\mu = rV_a\Delta t \cos\mu = sV_a \Delta t \cos^2\mu$; V_a = air speed, Δt = time increment, μ = Mach angle. One therefore has $A = (2\pi V_a \Delta t \cos^2\mu)s$ = (const) s for a uniform flight speed. For this case one can replace A_s/A by $1/s$ in Eq. (2.9), or in Eqs. (2.10) and (2.16) replace A by s. With this substitution Whitham's result, Eq. (44) of Ref. 4 can be obtained.

This result now will be extended to include cases for which the atmosphere is nonuniform and the aircraft may be accelerating. The rays will not be straight lines; however, they may be described by the equation $z = x \cot\nu$, as in Fig. 6. In this equation the shock angle, ν, varies along the ray path. The distance, d, of any point (x,z) to the ray $z = x \cot\nu$ is $d = z \sin\nu - x \cos\nu$. Consider, specifically, the distance to the ray $z = (x + e) \cot(\nu + \Delta\nu)$

$$d = z[\sin\nu - \cos\nu \tan(\nu + \Delta\nu)] + e \cos\nu \tag{4.1}$$

After letting $e = V_a \cos\theta \Delta t$, where $V_a \cos\theta$ is the air-speed component along the x axis (ray-coordinate system), one can expand Eq. (4.1) in powers of $\Delta\nu$ and Δt, obtaining as first-order terms

$$d = V_a \cos\theta \Delta t \cos\nu - z \sec\nu \Delta\nu \tag{4.2}$$

The increment $\Delta\nu$ can be related to increments in shock speed, winds, and aircraft speed by using Eq. (3.9). This leads to

$$\Delta\nu = \frac{\tan\nu}{V_s} \times \left\{ \Delta V_s - \sin\nu \left[\Delta V_a \cos\theta \left(\frac{M^2 \cos^2\theta - 1}{M^2 - 1} \right) + \Delta u_0 \right] \right\} \tag{4.3}$$

Using

$$\frac{\Delta V_s}{\Delta t} \approx \frac{dV_s}{dz}\frac{dz}{dt} \approx nV_s \frac{dV_s}{dz}$$

$$\frac{\Delta u_0}{\Delta t} \approx \frac{du_0}{dz}\frac{dz}{dt} \approx nV_s \frac{du_0}{dz}$$

and combining Eqs. (4.2) and (4.3), one obtains

$$d \approx \Delta t \left\{ V_a \cos\theta \cos\nu - \frac{z \tan\nu}{\cos\nu} \left[\frac{dV_s}{dz} n - \sin\nu \times \left[\frac{\cos\theta}{V_s} \frac{dV_a}{dt} \left(\frac{M^2 \cos^2\theta - 1}{M^2 - 1} \right) + \frac{du_0}{dz} n \right] \right] \right\}$$

Approximating the ray-tube area by $A = 2\pi |z| d$, and combining the equations below (4.3), one obtains finally

$$A \simeq \sec^2\nu_h |z| \left\{ \cos\nu + \frac{|z| \tan\nu}{V_a \cos\theta \cos\nu} \left[\frac{dV_s}{dz} n - \sin\nu \times \left[\frac{\cos\theta}{V_s} \frac{dV_a}{dt} \left(\frac{M^2 \cos^2\theta - 1}{M^2 - 1} \right) + n \frac{du_0}{dz} \right] \right] \right\} \quad (4.4)$$

The scaling constant, $\sec^2\nu_h$, has been inserted in order to have $A = s$ for a uniform atmosphere and flight speed.

If the aircraft is accelerating, the term dV_a/dt in Eq. (4.4) is positive. Neglecting the terms dV_s/dz and du_0/dz, one sees that for $|z|$ large enough the quantity within the braces vanishes. This gives rise to situations involving shocks of infinite amplitude as discussed by Rao.[7] In fact, if one sets

$$\theta = dV_s/dz = du_0/dz = 0$$

$$V_s = a_h = \text{sound speed at aircraft altitude}$$

$$(1/V_s)(dV_a/dt) = \dot{M}$$

$$\nu = \sin^{-1}(1/M) = \mu$$

$$z = s \cos\mu$$

one obtains Rao's result for an accelerating aircraft in a uniform atmosphere,

$$A = (\text{const})s[1 - (s\dot{M}/M^2 \cos^2\mu V_a)]$$

For the present theory, the term dV_s/dz is crucial. By using Rao's theory, the foregoing expression implies that for a certain value of s the area vanishes and the shock amplitude is infinite. The term dV_s/dz in Eq. (4.4) prevents this from happening. This is because as the shock strength increases it propagates faster and dV_s/dz increases; it will continue increasing until it counterbalances the negative contribution from the $-dV_a/dt$ term. From then on an equilibrium shock configuration is attained as it propagates.

Ordinarily, because of the $1/V_a$, all the terms in Eq. (4.4) are negligible in comparison to $\cos\nu$, but when complete acoustic refraction occurs $\cos\nu$ goes to zero. However, before this can happen the other terms in Eq. (4.4) increase in magnitude and the ray-tube area, A, is prevented from going to zero. Again an equilibrium shock configuration is attained.

It should be noted that in the shock strength Eq. (2.11), the area term is integrated with respect to ray distance s, whereas Eq. (4.4) gives the ray-tube area as a function of z. This can be resolved by simply replacing ds, in Eq. (2.11), by $(ds/dz)dz$ and using Eqs. (3.10) to evaluate ds/dz.

In closing this section an examination of the expression for ray-tube area, Eq. (4.4), and its derivation will be made. The shock refraction Eq. (3.9) relates the local shock angle with its initial angle. Hence the terms ΔV_s and Δu_0 in Eq. (4.3) can be interpreted as contributions to the change in shock angle from its initial angle as it moves along the ray. Similarly the term ΔV_a is the contribution to the change in shock angle as the aircraft moves along the flight path. The present theory therefore considers the initial ray angle as somewhat like an equilibrium position and that changes from this position are combined with the shock-strength and location equations, in a complicated manner, to determine a new, local equilibrium configuration.

5. Conclusions and Outline

5.1 Conclusions

The present theory, when used in its total generality, requires a simultaneous solution of the shock-strength equation (2.11) and the shock-location equations (3.10) with the ray-tube area being given in Eq. (4.4). By application of this theory, sonic boom intensity and location can be determined for arbitrary aircraft and atmospheric conditions. In addition, the theory can handle problems such as complete acoustic refraction and accelerating aircraft, which are beyond the scope of acoustic approximations.

When treating any specific problem, many simplifications could be made. For example, if one considered steady flight in which no acoustic refraction occurred, the ray-tube area could be simply approximated as $A \sim s$. Also, for most cases the acoustic ray-tracing equations (3.1) probably would provide sufficiently accurate shock-location data. However, the use of Eq. (2.11) for shock-strength determination should give, in all cases, a better answer than the isothermal-pressure correction.

At the present time a digital computer program is being written to carry out computations based on the present theory. Results of these computations and the evaluation of this theory will be given in a later note.

5.2 Summarizing Outline

A brief summary of the basic assumptions and equations will be given in this section. It is assumed that the aircraft altitude, flight pattern, and the conditions of the atmosphere are known. The objective is to locate the shock-ground intersection and to determine how the shock strength varies along this intersection. First, several angular positions around the initial aircraft Mach cone are chosen. Corresponding to each of these positions a ray is located, and the ray (x,y,z) coordinate system is defined relative to the aircraft (X,Y,Z) axes. This procedure is described in Sec. 3.1. The next step is to determine the path of shock propagation, using ray-tracing equations (3.10). These equations, derived in the Appendix, are improved in Sec. 3.2 to account for actual shock propagation speed. However, in order to determine shock-propagation speeds the shock strength must be known. The variation in shock strength, as it propagates along the ray, is determined in Sec. 2 and is given in terms of pressure jump in Eq. (2.16). Propagation speed and pressure jump are related in Eq. (3.12). An important factor in determining the shock strength is the ray-tube area, and this is discussed in Sec. 4. Therefore, the ray intersection and the shock-strength variation at the ground are determined by integrating the ray-tracing equations (3.10) in conjunction with the shock-strength equations (2.16) and (3.12), and the ray-tube area equation (4.4). A technique for putting this in terms of the shock-ground intersection is given in Sec. 3.3.

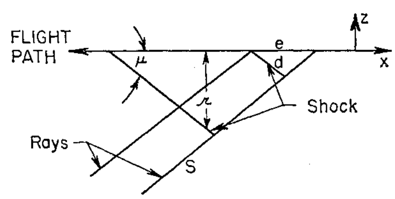

Fig. 5 Ray-tube coordinates for a uniform atmosphere

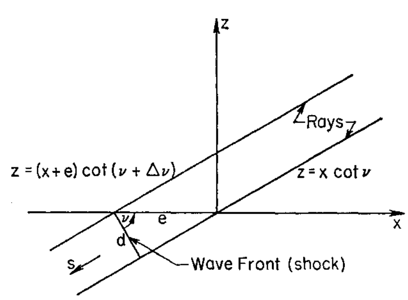

Fig. 6 Generalized ray tube coordinates

Appendix

A.1 Derivation of Acoustic Ray-Tracing Equations

All equations appearing in Secs. A.1 and A.2 are written relative to a fixed coordinate system. The transformation to the moving coordinate system, used in Sec. 3, is given in A.3. In addition, the repeated subscript summation convention will be used in order to shorten the presentation.

The equation for the acoustic wave front can be derived as a characteristic of the Eulerian flow equations. However, a simple derivation is possible if one starts with the statement: the acoustic wave front travels in a direction normal to its surface, at sound speed relative to its ambient atmosphere.

If the wave front is denoted by $\phi(x,y,z,t) = \phi(x_i,t) = 0$ the direction cosines of the normal to the front are

$$n_i = \phi_{xi}/(\phi^2_{xi})^{1/2} = \phi_{xi}/Q \tag{A1}$$

A point x_i, on the surface at time t, will at time $t + \Delta t$ be at $x_i + n_i\Delta N$, where ΔN is the perpendicular distance between the surfaces. Since x_i and $x_i + n_i\Delta N$ are on the surface

$$\phi(x_i, t) = 0 \tag{A2}$$

$$\phi(x_i + n_i\Delta N, t + \Delta t) = 0 \tag{A3}$$

Expanding Eq. (A3) in a Taylor series about the point (x_i,t) one has, after using Eq. (A2) and retaining first-order terms,

$$\Delta N n_i \phi_{xi} + \Delta t \phi_t = 0$$

Divide by Δt, and then let Δt go to zero to obtain the surface normal velocity $(dN/dt) = -(\phi_t/Q)$. The components of this velocity along the coordinate axes are

$$(dx_i/dt) = -(n_i\phi_t/Q) \tag{A4}$$

These velocity components relative to the wind components, u_i, are

$$(dx_i/dt) - u_i \tag{A5}$$

If the velocity components, Eq. (A5), are projected onto the surface normal one has, from the definition of the velocity of an acoustic wave front,

$$[(dx_i/dt) - u_i]n_i = +a \tag{A6}$$

or

$$\phi_t + u_i\phi_{xi} + aQ = 0$$

Equation (A6) describes a wave front moving through the atmosphere; one sees that the velocity components, Eq. (A5), must satisfy the relation

$$(dx_i/dt) - u_i = n_i a \qquad i = 1,2,3 \tag{A7}$$

These three equations give the velocity of a point, x_i, on the front; the locus of this point, as the surface moves through space, is called a ray. Equation (A7) shows that, if there is no wind, the ray is normal to the surface of the front. In order to solve Eqs. (A7) the direction cosines n_i must be determined. These, however, vary with the atmosphere as one moves along the ray; the differential equation for this variation will be determined now.

Letting d/dt denote differentiation along the ray one has, using Eq. (A1),

$$\frac{dn_i}{dt} = \frac{1}{Q}\frac{d}{dt}\phi_{xi} - \frac{n_i n_j}{Q}\frac{d}{dt}\phi_{xj} \tag{A8}$$

where

$$\frac{d}{dt}\phi_{xi} = \frac{\partial}{\partial t}\phi_{xi} + \frac{dx_k}{dt}\frac{\partial}{\partial x_k}\phi_{xi} = \frac{\partial}{\partial t}\phi_{xi} + (u_k + n_k a)\frac{\partial}{\partial x_k}\phi_{xi}$$

Differentiating the second equation in (A6) are combining the result with Eq. (A8) one obtains

$$\frac{1}{n_i}\left(\frac{dn_i}{dt} + n_k\frac{\partial u_k}{\partial x_i} + \frac{\partial a}{\partial x_i}\right) = n_j n_k \frac{\partial u_k}{\partial x_j} + n_j\frac{\partial a}{\partial x_j} \tag{A9}$$

The right-hand side is independent of subscript i and is the same for each n_i, $i = 1,2,3$; therefore the differential equation for the direction cosines of the surface normal can be written as

$$\frac{1}{n_1}\left(\frac{dn_1}{dt} + n_k\frac{\partial u_k}{\partial x_1} + \frac{\partial a}{\partial x_1}\right) = \frac{1}{n_2}\left(\frac{dn_2}{dt} + n_k\frac{\partial u_k}{\partial x_2} + \frac{\partial a}{\partial x_2}\right) = \frac{1}{n_3}\left(\frac{dn_3}{dt} + n_k\frac{\partial u_k}{\partial x_3} + \frac{\partial a}{\partial x_3}\right) \tag{A10}$$

Equations (A7), (A10), and $n_i^2 = 1$ are six equations for the six unknowns x_i,n_i along the ray.

These equations now will be simplified. First make the identification

$$(x_1,x_2,x_3,n_1,n_2,n_3) = (x,y,z;l,m,n)$$

Now assume the cross winds (u,v) and sound speed a to be dependent only on altitude z; also, vertical winds are to be neglected. The ray equations now become

$$\frac{dx}{dt} = la + u \qquad \frac{dy}{dt} = ma + v \qquad \frac{dz}{dt} = na$$

$$l^2 + m^2 + n^2 = 1 \tag{A11}$$

$$\frac{1}{l}\frac{dl}{dt} = \frac{1}{m}\frac{dm}{dt} = \frac{1}{n}\left(\frac{dn}{dt} + \frac{da}{dz} + l\frac{du}{dz} + m\frac{dv}{dz}\right)$$

From a solution of the first equality in the last equation, $l/l_h = m/m_h$ with l_h and m_h initial direction cosines. The z axis has been set as being vertical, however one is still at liberty as to the direction of the horizontal x,y axes. Let the x axis be so positioned that the normal to the wave front is parallel to the x,z plane; then $m = m_h = 0$ (The details of this coordinate positioning are given in Sec. 3.1.) Equations (A11) now read

$$\frac{dx}{dt} = la + u \qquad \frac{dy}{dt} = v \qquad \frac{dz}{dt} = na$$

$$l^2 + n^2 = 1 \qquad \frac{dl}{dt} = \frac{l}{n}\left(\frac{dn}{dt} + \frac{da}{dz} + l\frac{du}{dz}\right) \tag{A12}$$

The last three equations in (A12) can be combined to give

$$\frac{dl}{dz} = \frac{l}{a}\left(\frac{da}{dz} + l\frac{du}{dz}\right)$$

which integrates to

$$(a/l) + u = (a_h/l_h) + u_h = \text{const} \tag{A13}$$

This is Snell's law for a varying atmosphere, the right-hand side being specified by initial conditions. With Eq. (A13) and $l^2 + n^2 = 1$, one has integrated the direction-cosine equations. The ray equations are written now in their final form:

$$\frac{dx}{dz} = \frac{la + u}{na} \qquad \frac{dy}{dz} = \frac{v}{na} \qquad \frac{dt}{dz} = \frac{1}{na}$$

$$\frac{ds}{dz} = -\left[\left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2 + 1\right]^{1/2} \tag{A14}$$

$$\frac{a}{l} + u = c\ (\text{const}) \qquad l^2 + n^2 = 1$$

It is assumed that a and u are known as functions of altitude

z. Equation (A14) can be integrated from aircraft altitude to ground to give the point of intersection of the ray and the ground and the time it takes the ray to reach the ground.

A.2 Atmospheric Refraction

Complete acoustical refraction occurs when an initially downward traveling ray bends upward. The cause of this phenomenon now will be discussed. At the point of horizontal slope $n = 0$ and, hence,

$$(c - u)^2 - a^2 = (c - u - a)(c - u + a) = 0 \qquad \text{(A15)}$$

Assume that the ray moves downward in the negative x direction (see Fig. 5); then both l and n are negative, and the constant c is also negative, as for all practical cases $a > u$. Hence Eq. (A15) can vanish only when $c - u + a$ vanishes. Now consider the ray directly below the flight path; for this case $l_h = -\cos(90° - \mu) = -\sin\mu = -a_h/|V_a|$ where μ is Mach angle and V_a is aircraft air speed. When the above results are combined one sees that the ray will bend upward if $-|V_a| + a + (u_h - u)$ vanishes, i.e., if $a + (u_h - u)$ increases sufficiently as the ray travels downward. A headwind decreasing or sound speed increasing as the ground is approached will cause a ray to be bent upward. Conversely, tail winds decreasing groundward will bend rays downward.

A.3 Transformation to Moving Coordinate System

As mentioned at the beginning of Sec. 2, it is necessary to measure shock-travel distance and atmospheric wind variations relative to a coordinate system moving with the wind at aircraft altitude. In this section relations between the fixed coordinate system used in the prior two sections, and the moving coordinate system will be developed. In addition, ray-tracing equations, corresponding to (A12), will be derived for the moving coordinate system.

Denote the fixed system by (x_f,y_f,z_f), the moving system (x,y,z), and the wind components along the x_f,y_f axes by u_h,v_h, respectively. The two systems are related as follows:

$$\begin{aligned} x_f &= x + u_h t \\ y_f &= y + v_h t \qquad \text{(A16)} \\ z_f &= z \end{aligned}$$

Consider, now, Eqs. (A12) (a subscript f should be affixed to the coordinates appearing there) and substitute (A16):

$$\frac{dx}{dt} = la + u - u_h \qquad \frac{dy}{dt} = v - v_h \qquad \frac{dz}{dt} = na$$

$$l^2 + n^2 = 1 \qquad \frac{dl}{dt} = \frac{l}{n}\left(\frac{dn}{dt} + \frac{da}{dz} + l\frac{du}{dz}\right) \qquad \text{(A17)}$$

If one now introduces in (A17) wind components relative to the wind at aircraft altitude, u_0,v_0: $u_0 = u - u_h$; $v_0 = v + v_h$, the equation takes exactly the same form as (A12). Snell's law, (A13), can be written as

$$(a/l) + u - u_h = (a/l) + u_0 = (a_h/l_h) = c \qquad \text{(A18)}$$

Therefore, relative to a coordinate system moving with the wind at aircraft altitude, the ray-tracing equations are as shown in Eqs. (3.1).

References

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